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On the 'wave momentum' myth

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Controversies over 'the momentum' of waves have repeatedly wasted the time of physicists for over half a century. The persistence of the controversies is surprising, since regardless of whether classical or quantum dynamics is used the facts of the matter are simple and unequivocal, are well checked by laboratory experiment, are clearly explained in several published papers, and on the theoretical side can easily be verified by straightforward calculations. They are illustrated here by some simple, classical examples involving acoustic and gravity waves.

'The question is,' said Alice, 'whether you *can* make words mean different things.'
'The question is,' said Humpty Dumpty, 'which is to be master – that's all.'

– Carroll (1871)

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1. Introduction

The privilege of writing an article for this special issue of *JFM* gives one, to quote Bondi (1967), 'a rare opportunity to allow the bees in one's bonnet to buzz even more noisily than usual'. One of the oldest and noisiest, in my case, concerns the myth that waves possess momentum. Most readers of *JFM* will know that waves in fluids do not generally have a uniquely defined mean momentum – surface gravity waves being an exception, in a certain sense, in this as in so many other ways. But in the literature on other types of waves one still comes across the catch-phrases that propagate the myth. Waves are said, for instance, to give up 'their momentum' near a critical layer, exchange 'their momentum' with the mean flow, and so on. Often what is actually referred to (as evidenced by the equations written down) is the convergence of a wave-induced stress, or *flux* of momentum.

Such seemingly harmless licence in the use of words is best viewed in the light of history. A foray into the literature of physics (good entry points being the papers by Post (1953), Gordon (1973), Dewar (1977) and Israel (1977)) reveals a long and surprising history of fallacy, misunderstanding and controversy over 'the momentum' of various kinds of waves. To quote E. I. Blount, 'The argument has not, it is true, been carried on at high volume, but the list of disputants is very distinguished' (unpublished manuscript cited by Gordon, *op. cit.*). The list includes Rayleigh, Poynting, Ehrenfest, Brillouin, Abraham, Minkowski, and Pauli. The controversies continue today, and are the more surprising since the questions involved are simple ones of classical physics. The persistence of the controversies is partly due, I believe, to the persistence of verbal inaccuracies of just the sort cited above. The facts of the matter can easily be put into perspective by taking a look at three simple examples involving ordinary acoustic and gravity waves.

2. Acoustic waves in a tube

It was symptomatic of the state of affairs in 1925 that L. Brillouin felt the same need to sound off about the fact that momentum density and momentum flux are independent entities. In his classic paper on radiation stress in acoustic and elastic waves, probably the first clear presentation of the idea of radiation stress (Brillouin 1925), he wrote:

It is not ultimately the density of momentum which matters, but rather the *flux of momentum*. This latter may very well differ from zero even when the density of momentum is zero. [My translation, but his italics.]

Brillouin was reacting to a mistake in Rayleigh's pioneering writings on the subject. Rayleigh (1905) had endorsed a statement by Poynting about the wave-induced mean force on a reflector, to the effect that

if the reflexion of a train of waves exercises a pressure upon the reflector, it can only be because the train of waves itself involves momentum.

By implication the statement just quoted, if true, would apply also to a wave emitter or absorber. But, as Brillouin pointed out, the deduction that the waves involve momentum is entirely fallacious. Fluxes of momentum (i.e. stresses, apart from sign), can perfectly well exist in a material medium without there being any momentum present. This of course is one of the basic differences between waves in media and waves *in vacuo*. To be sure, the distinction between vacuum and medium was not entirely clear in 1905 (if we discount an unknown researcher by the name of A. Einstein). But Rayleigh's mistake is still a surprising one, not only because Rayleigh seldom made mistakes, but also because one does not have to look far to find counter-examples.

The simplest is one of those considered by Brillouin. One-dimensional, plane, progressive acoustic waves in an inviscid fluid are imagined to be propagating down a tube, being emitted by a vibrating piston at one end and perfectly absorbed by a similarly vibrating piston at the other. In the steady state, with each piston vibrating about a fixed mean position, it is obvious from mass conservation that the centre of mass of the fluid cannot have any systematic mean motion. Each fluid particle must likewise be oscillating about a fixed mean position. Therefore no mean momentum can be present. There is certainly, on the other hand, a wave-induced *flux* of momentum along the tube, together with corresponding mean forces on the emitter and absorber. As is well known, these are given for small wave amplitude a by

$$\hat{E} + \hat{E}\frac{\partial(\log c)}{\partial(\log \rho)} + O(a^3) \qquad (2.1a)$$

per unit cross-sectional area, where $\hat{E}$ is the usual acoustic energy density, a positive definite, $O(a^2)$ quantity. The derivative of the sound speed c with respect to density ρ is the adiabatic derivative. The expression (2.1a) gives the excess mean inward force on each piston, per unit area, required to maintain its mean position while vibrating.

In other words, if the fluid is initially at rest and the pistons are started vibrating in an appropriately smooth manner, (2.1 *a*) is the increase in mean inward force per unit area on each piston required to keep the mean volume constant. It differs from the excess mean force per unit area on the side of the tube, which is given by

$$\hat{E}\,\frac{\partial(\log c)}{\partial(\log \rho)}+O(a^3). \tag{2.1b}$$

The quantity $\partial(\log c)/\partial(\log \rho)$ is positive for most fluids. The expressions (2.1) show that the waves set up an anisotropic mean stress within the fluid, which tries to push the ends of the tube apart more strongly than the sides. Brillouin called this the radiation stress.

It is worth noting that at least four independent derivations of (2.1) are available, whose results are in complete agreement. This is emphasized because (2.1) has been questioned a number of times in the acoustics literature. The usual mistake is to solve the one-dimensional, nonlinear initial-value problem for emission of sound into a semi-infinite tube, by Riemann's or Airy's methods (e.g. Fubini-Ghiron 1937), but neglect to notice that the mean volume of a given mass of fluid in the tube generally suffers a permanent, $O(a^2)$ change when the waves arrive. Choice of solution parameters such that the mean volume has its stipulated value *after* the waves have arrived does reproduce (2.1); the nonlinear solutions also verify, of course, that each fluid particle does oscillate about a fixed mean position. A second derivation of (2.1) may be constructed by posing expansions in powers of a from the outset, with due care over boundary conditions. A third follows directly from Lagrangian-mean theories of wave, mean-flow interaction (Bretherton 1971, § 6.5; Andrews & McIntyre 1978*a*, § 8.4)). A fourth derivation appeals to the Boltzmann–Ehrenfest theorem of 'adiabatic invariance', which states that the energy of a harmonic oscillator, vibrating freely and subject to slowly varying constraints, is proportional to its frequency. Use of the Boltzmann–Ehrenfest theorem permits the $O(a^2)$ mean forces felt by the constraints to be deduced from a knowledge of the oscillations to $O(a)$ only. Brillouin did this in the corresponding standing-wave problem in which the pistons do not vibrate (but can be imagined to be moved slowly in or out). As I show elsewhere (McIntyre 1981), a modicum of ingenuity allows the theorem to be applied directly to the progressive case as well.

The derivation from the Boltzmann–Ehrenfest theorem extends immediately to longitudinal dispersive waves (e.g. Post 1953) for which the dispersion relation does not itself depend on direction of propagation. The result is the same as (2.1) except that the anisotropic contribution $\hat{E}$ to the radiation stress, the first term in (2.1 *a*), is replaced by

$$\hat{E}C/c, \tag{2.2}$$

where C is the group velocity. The expression (2.2) was obtained by Rayleigh (1902) using *ab initio* expansion in powers of a. This contribution to the wave-induced momentum flux is the same *as if* the medium were absent, and progressive waves possessed mean momentum $\hat{E}/c$. The words 'as if' are crucial, unless the medium really is absent. In the case of waves in a vacuum, light waves for instance, wave-induced mean stresses can, of course, be attributed to the propagation through space of a disturbance which possesses momentum. But waves in media are fundamentally different from waves in a vacuum, as already suggested. The difference is truly a fundamental one: for instance the linear equations describing waves in a vacuum satisfy the principle of relativity, whereas in the presence of a material medium they do not, because the medium defines a special frame of reference.†

3. The 'towing' experiment

There is a celebrated thought-experiment in which waves are generated by towing a rigid, slender, body through an inviscid fluid initially at rest. The idea goes back a long way in the history of the subject (Lamb 1932, § 249) and of course arises immediately in studies of wave generation by ships, submarines, and supersonic aircraft. As is well known, the wave-induced drag D on such a slender body is calculable correct to $O(a^2)$ from linear theory. This is connected with the fact that the wave-energy $\hat{\mathscr{E}}$ (the volume integral of $\hat{E}$) is so calculable. The rate of increase of $\hat{\mathscr{E}}$ is equal to the rate at which the agency propelling the body against the wave drag does work. Thus

$$d\hat{\mathscr{E}}/dt = UD,$$

where U is the speed of the body through the fluid, which together with D we shall take to be constant. If I is the wave drag multiplied by the time elapsed, we then have

$$I = \hat{\mathscr{E}}/U. \tag{3.1}$$

It has sometimes been argued that because the $O(a^2)$ quantity I can be calculated from linear theory it ought to represent a momentum associated with the waves. The argument is fallacious, as before, except for waves in a vacuum. The most that is implied by (3.1) is that the total momentum M of the system has undergone a change equal to I as the result of wave generation by towing, assuming that M is well defined and that no forces other than D are exerted on the system. There is no general reason, for instance, why the added momentum should be distributed spatially in the same way as the waves themselves.

Explicit calculations have shown that the spatial distributions are indeed very different in many cases. I shall now describe one such case, in which the momentum given to the system is well defined and equal to I, but is distributed so differently from the wave-energy generated by the towed body as to leave no room for doubt about the facts of the matter.

Consider a two-dimensional version of the towing experiment in which all quantities are independent of y, and the body is towed horizontally along a waveguide consisting of stably stratified fluid lying between rigid, horizontal boundaries $z = 0, h$, as suggested in figure 1 (a). This is how internal gravity waves are often generated in laboratory tanks, except that here we are assuming an infinitely long tank and the fluid

† If we were including relativistic effects *per se*, incidentally, in the sense of Einstein's special theory, then the energy flux $\hat{E}C$ in our mechanical waves would imply that a minute but non-zero mean momentum density, equal to $\hat{E}C$ divided by the square of the speed of *light*, must be present (Landau & Lifshitz 1959, chapter 15). Its physical origin lies in several effects, one of which is the relativistic increase in the mass of a material fluid element when its pressure and internal energy rises; in a progressive sound wave a fluid element is more massive while it is moving forward. The mean forward motion of the centre of mass of the whole system is accounted for by the energy and therefore mass depletion of the agency emitting the waves, and the corresponding mass increase due to the work done on the wave absorber at the other end of the tube.

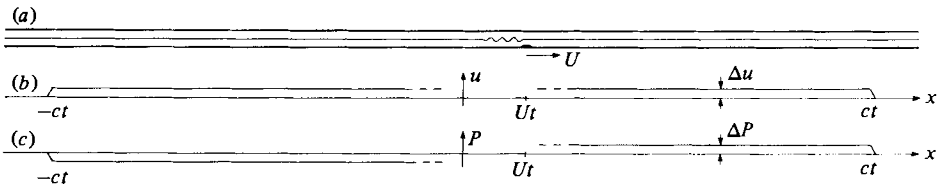

FIGURE 1. (*a*) Sketch of internal gravity waves being generated by a two-dimensional, towed body in a waveguide containing a stratified fluid. The central 'dye streak' marks the location of the gravity waves. (*b*), (*c*) Schematic distributions of $O(a^2)$ velocity and relative pressure, well outside the region containing gravity waves. Details near the latter region are omitted, as are starting transients affecting the profiles near $x = \pm ct$. The graphs show the outer solutions only, which are the same at all levels z if the slight variation of sound speed c and acoustic impedance with depth is ignored.

is unbounded in the x direction. For suitable values of the speed U a lengthening train of internal gravity waves appears behind the body.† For an incompressible fluid, the velocity field satisfies

$$\operatorname{div} \mathbf{u} = u_x + w_z = 0.$$

We take w to be zero on the top and bottom boundaries, and so

$$\frac{\partial}{\partial x}\int_0^h u\,dz = 0 \tag{3.2}$$

for any value of horizontal position x not coinciding with the body. This expresses the fact that the volume flux of an incompressible fluid cannot change along the waveguide. Let us for the moment make the Boussinesq approximation, which assumes not only incompressibility but also that variations in fluid inertia per unit volume can be neglected (so that density variations are considered significant only when multiplied by the acceleration due to gravity). Then (3.2) implies that the density of horizontal momentum per unit horizontal area is constant as x varies, to the extent that the approximation holds. This makes it evident at once that the momentum of the fluid cannot be spatially distributed in the same way as the waves, with the possible exception of a small contribution whose magnitude depends on the accuracy of the Boussinesq approximation and therefore has no simple relationship to (3.1).

Incompressibility fails at distances of the order of the sound speed c multiplied by the time t elapsed since the thought-experiment commenced. Since our assumptions imply a very large sound speed $c \gg U$, most of the momentum must be located far away from the gravity waves. It is a simple exercise in matched asymptotic expansions to show that the momentum appears in the form of one-dimensional, $O(a^2)$ acoustic disturbances with velocity and pressure distributions like those depicted schematically in figs. 1 (*b*), (*c*). The pressure shown is the excess pressure relative to the undisturbed, hydrostatic pressure at each level z. We have ignored the effects of starting transients, for instance those due to the initial acceleration of the body and the associated inertial reaction of the fluid.

† The detailed linear theory can be found in my (1972) paper and follows much the same pattern as well-known calculations for other kinds of dispersive waves. It ignores certain non-linear, essentially transient effects near the body observed in actual laboratory experiments in some parameter ranges (Baines 1979), and assumes uniform stratification.

Within our approximations it is easy to check directly that the velocity distribution associated with these long acoustic pulses does contribute a momentum just equal to I. The far ends of the pulses can propagate freely only if the pressure change ΔP and velocity change Δu are in a constant ratio, equal in magnitude to the acoustic impedance ρc of the fluid and having the same sign as x on either side, i.e.

$$\Delta P = \pm \rho c \Delta u \quad (x \gtrless 0). \tag{3.3}$$

The fact that (3.2) holds across the inner region containing the gravity waves implies that the values of Δu are the same at both ends; therefore the values of ΔP are equal and opposite. To fix the magnitudes of ΔP and Δu, one more relation is needed. It comes from the fact, easily verified from (3.2) and the equation of motion, that the pressure drop across the inner region must equal h^{-1} times the steady force D per unit width applied to that region, i.e.

$$2|\Delta P| = h^{-1}D. \tag{3.4}$$

It is here that we neglect details associated with starting transients. Solving (3.3) and (3.4) for Δu, we have

$$u = \Delta u = D/2\rho hc$$

outside the region containing gravity waves but within the sound pulses. This $O(a^2)$ velocity tends to zero if we let c tend to infinity. But since the acoustic pulses extend over a distance $2ct$, their contribution to the total momentum per unit width of the system is

$$2ct\,\rho h\,\Delta u = tD, \tag{3.5}$$

which is independent of c, and equal to I.

If the body is brought to a halt so that the train of $O(a)$ gravity waves becomes a freely propagating wave packet, the $O(a^2)$ acoustic pulses become completely detached from the region containing gravity waves. This follows from the same considerations as before, but with $D = 0$ in (3.4). Thus the point being illustrated (that the waves and the momentum are unconnected with each other, except insofar as they are produced simultaneously by one particular way of generating the waves) evidently holds quite independently of any particular definitions which might be used, for instance, to distinguish 'waves' from 'mean state'.

So far it has been assumed that no horizontal forces other than D are exerted on the fluid. But if we were to imagine the experiment done in a tank of finite length L, less than ct for the values of t of interest, then the forces exerted on the fluid by the end walls would change the picture completely. If the tank is mounted rigidly to the same support (e.g. the laboratory floor) from which the force towing the obstacle is exerted, then no net momentum need appear at all. We may suppose if we wish that the towing is started smoothly over a time $t \gg L/c$, in which case acoustic disturbances can be shown to be altogether negligible. Alternatively, strict incompressibility of the fluid could be assumed at the outset, as is often done as a convenient basis for model calculations of gravity waves. It is often done, indeed, for the case of a completely unbounded fluid. The inconsistency inherent in the idea of an unbounded yet incompressible fluid gives no trouble *except* when we try to discuss the total momentum and the pressure reaction at infinity (the forces exerted by the walls, so to speak, of an infinitely large tank). These are given by divergent integrals, as is well known (e.g.

Batchelor 1967). In the case of an unbounded, stratified fluid the same difficulty arises with the 'fluid impulse' as well (Andrews 1974).

The fluid-impulse concept does, however, apply to the incompressible, inner region in figure 1(*a*), as was pointed out by Benjamin (1970). For an incompressible fluid the impulse is an integral property of the fluid motion, involving the vorticity, whose rate of change equals D regardless of pressure changes at large x. In my (1972, 1973) papers I used an explicit solution for the inner region, correct to $O(a^2)$, to evaluate the integral giving the impulse. It is equal to I of course, but the interesting aspect of the calculation is that the integrand turns out to have a distribution in x which differs both from that of the $O(a^2)$ momentum and from that of the $O(a)$ gravity waves – as can be seen at a glance from figure 2 (*a*) of the (1973) paper. The same calculation shows, incidentally, that there will be a small contribution to the total mean momentum arising from the variation of inertial density of the fluid due to the stable stratification. This is the effect previously neglected through appeal to the Boussinesq approximation for the region containing gravity waves. The non-Boussinesq contribution is much smaller than those discussed so far, and its spatial distribution once again differs both from that of the $O(a^2)$ acoustic pulses in figure 1 (*b*) and from that of the $O(a)$ gravity waves in figure 1(*a*).

If momentum is not a general property of waves, one might well ask what is meant by the term 'wave momentum' sometimes encountered in the literature on general wave theory, and why there is a significant class of problems in which the right answer can be obtained by pretending that the fluid is absent and that the waves possess momentum given by formulae like (3.1). I shall answer this question in section 5. But part of the answer is already hinted at clearly enough by the present thought-experiment, as well as by Brillouin's remarks quoted earlier. The wave-drag D represents a force exerted across a material surface, equivalent to a *flux* of momentum. It is to fluxes of momentum, and not to momentum itself, that relations like (3.1) are directly relevant in general.

4. The case of surface gravity waves

Because of the connection just referred to, between wave drag and wave-induced momentum flux, there are a few special cases in which mean momentum equal to I not only appears in the fluid but does seem to accompany the waves. The simplest examples are those in which such a result is inevitable because the geometry assumed permits nothing else to happen. Consider for instance a version of the internal-gravity-wave problem of figure 1 (*a*) in which the towed obstacle is replaced by a sinusoidally corrugated, moving boundary extending to infinity in both the positive and negative x directions. If the boundary moves at a phase speed matching that of a free gravity mode then there is a resonant buildup of wave-energy, and an associated wave drag. If we assume that pressures remain finite at $x = \pm\infty$ so that no mean pressure gradients in the x direction arise, then an amount of momentum equal to I is given to the fluid as before, reckoned now per unit distance in x as well as in y. The momentum is spatially coincident with the waves simply because there is no other possibility: the waves are generated everywhere simultaneously.

Perhaps the most famous example of this sort is the case of strictly periodic surface gravity waves imagined to have been generated from rest, in an inviscid, incompressible fluid of large depth H, by a travelling, periodic surface pressure distribution. The motion is irrotational if the fluid is unstratified, and this leads to a special result of some interest, namely that all the momentum appears near the surface in the form of the Stokes drift. The Stokes drift is defined generally as the difference between mean particle velocity and Eulerian-mean velocity. It is a wave property, calculable from linear wave solutions, and as is well known is not generally zero. The fact that it accounts for *all* the mean momentum in this problem is attributable to the fact that in the absence of mean horizontal pressure gradients the Eulerian-mean velocity cannot change during wave generation, being irrotational and independent of x, and therefore of z. In the periodic internal-gravity-wave problem, by contrast, the motion is rotational, the Eulerian-mean velocity does change during wave generation, and the Stokes drift does not account for the total momentum.

Because the Stokes drift is a wave property, it might be thought that spatial coincidence of the mean momentum with the waves themselves would carry over to wavetrains of finite length, in the case of surface gravity waves. That is true, however, only in the superficial sense that the part of the mean momentum associated with the Stokes drift does, by definition, stay with the waves. For finite wavetrains there are always further contributions to the momentum of the system. In a fluid of depth H, for instance, the two-dimensional problem of wave generation from a finite forcing region has much the same character as in figure 1. The problem has been discussed by Benjamin (1970) for the case where the waves are generated by a towed body. In place of the $O(a^2)$ acoustic pulses in figure 1 there are slower-propagating, $O(a^2)$ changes in the level of the free surface. After a sufficiently long time these are distributed qualitatively as in figure 1 (c), the relevant propagation speed c now taking the value

$$c = (gH)^{\frac{1}{2}},$$

the speed of long surface gravity waves. The associated horizontal velocities are distributed qualitatively as in figure 1 (b). The mass flux of the Stokes drift recirculates in a return flow underneath the wavetrain, superposed on the other $O(a^2)$ motions.

If we let H tend to infinity in this problem, the depth-integrated momentum appears at distances $|x|$ further and further away from the wavetrain, just as it did in the problem of figure 1 when we let the sound speed tend to infinity. The details change somewhat as soon as H exceeds ct, i.e.

$$H \gtrsim gt^2,$$

but the conclusion is unaffected, essentially because propagation speeds for net changes in surface level still increase without bound as H increases. (Indeed it turns out that the depth-integrated momentum spreads further than such considerations might suggest, occupying a horizontal distance of order $H \gg ct$ as $H \to \infty$ for given t.) As in the example of figure 1, the magnitudes of the net changes in surface level and associated fluid velocities tend to zero in the limit, but the total momentum associated with them does not.

If wave generation ceases then the propagating surface-level changes move off to either side, as in the internal-gravity-wave example, and we are left with the state of affairs suggested by figure 2. The Stokes drift has momentum equal to I, and the deep, irrotational return flow has momentum equal to $-I$. If the wavetrain can be considered to be slowly-varying, and if H is large compared to the wavelength, then there is no further contribution to the local $O(a^2)$ mean motion (Longuet-

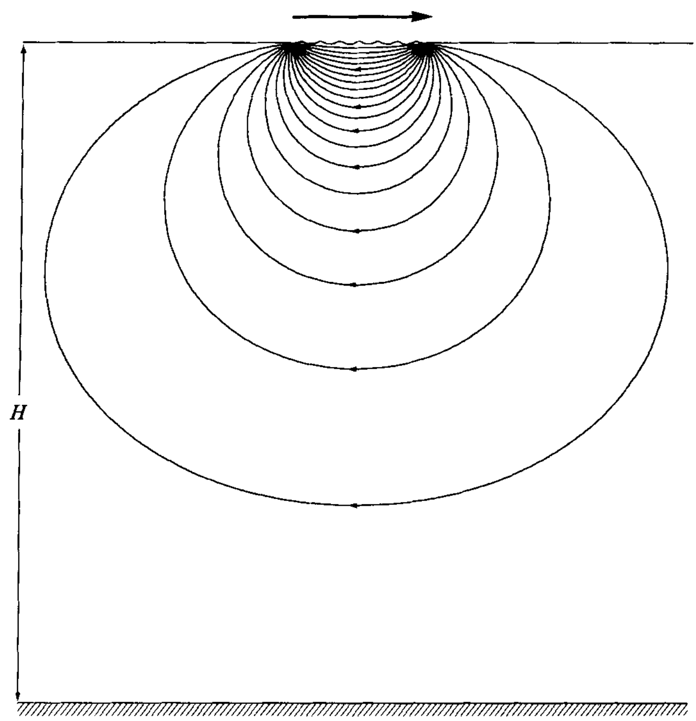

FIGURE 2. The irrotational, $O(a^2)$ return flow underneath a packet of surface gravity waves propagating to the right. (The streamlines, plotted at equal intervals, are quantitatively correct for a two-dimensional wave packet whose amplitude is constant except near its ends.)

Higgins & Stewart 1962, equation (3.26), and (3.18) with $\theta kH \gg 1$). The reason is that the surface appears rigid to the local mean flow under the assumed conditions, since the group velocity is much slower than propagation speeds for surface-level changes on the scale of the whole wave-train. In that case the third contribution $+I$, needed to bring the total momentum back up to its known value I, resides entirely in the propagating contributions, which as time goes on are found at increasingly remote distances $|x|$ to the left and right of the picture. It is the first of the three contributions, of course, which is usually referred to as 'the wave momentum' in the literature on surface gravity waves. It is interesting that the second contribution, associated with the return flow in figure 2, has been shown to play a significant role in the dynamics of modulational instabilities of surface gravity waves (Dysthe 1979). It should perhaps be added also that in real oceanographic applications the existence of stable stratification can greatly modify the form of the return flow, coupling it directly to internal gravity waves.

Not all examples of isolated wave packets have zero momentum like this one. If H is not much greater than the length of the wavetrain, then an isolated packet of surface gravity waves has finite momentum, not equal to I (Longuet-Higgins & Stewart *op. cit.*). An isolated packet of sound waves does have momentum equal to I (Landau & Lifshitz 1959), but the momentum is not distributed spatially in the same way as the $O(a)$ waves; indeed its distribution varies secularly in time, as one might have anticipated from the theory of finite-amplitude sound waves. An isolated packet of light waves in a refractive medium has a well-defined momentum in some simple cases. The momentum depends on the shape of the wave packet and the sound speed, and is not equal to I in general (Gordon 1973; Robinson 1975; Peierls 1976; Dewar 1977). Other cases are known in which the packet continuously radiates an $O(a^2)$ disturbance, and does not have a well-defined momentum at all (e.g. Bretherton 1969, § 3.3; Peierls 1976, § 7).

5. The pseudomomentum rule

It will have been noticed that I have been using the terms 'energy' and 'momentum' in the ordinary, elementary sense of Newtonian mechanics. It is important to remember that some of the more abstract ways in which such terms are used actually contradict their elementary physical meanings. This has come about because in mathematical thinking it is customary and natural to use terms, whose original motivation was physical, to refer to mathematical generalizations of the original concepts, with no attention paid to physical meaning. Confusion can arise when such a mathematical concept is mistaken for the physical one which originally suggested it. This mistake – which appears to be one of those underlying the controversies about wave 'momentum' – is like the more obvious one of saying that an irrotational fluid motion is the same thing as an electrostatic field. Insofar as they both satisfy Laplace's equation they are certainly the same thing mathematically, but no one would be likely to argue that they are the same thing physically.

The basic properties which make quantities like energy and momentum useful are their conservation properties; and the basic general remark about conservation relations is their well-known correspondence with symmetry operations. Conservation of total energy, for instance, is related to a certain type of symmetry with respect to time. More precisely, energy conservation requires that the laws governing the motion of a given dynamical system be independent of time, including any external constraints on the system. A different symmetry condition, which may be of interest when waves propagate in a material medium, is that the *medium* be time-independent. The corresponding conservation relation may be called conservation of pseudoenergy, following precedents in plasma physics (e.g. Sturrock 1962*b*). The medium can be stationary or moving, in the chosen frame of reference, but if pseudoenergy is to be conserved then any mean flow which is present must be steady. From a physical point of view time-independence of the medium is not, of course, the same thing as time-independence of the laws of motion and external constraints. The latter condition concerns the whole system, waves and mean flow included. The former concerns the problem for the waves alone, which in its simplest form is just the linearized problem. Consequently pseudoenergy is (*a*) a wave property, evaluable from a linear wave solution, and (*b*) is not physically the same thing as energy. It may of course be of interest to enquire whether the two entities are related in any way. It can be shown (McIntyre 1981) that there is no general relation between their densities, but that there is, on the other hand, a fairly close general relation (although not equality) between their fluxes.

Similar considerations apply to conservation relations associated with spatial symmetries. Translational invariance of the whole dynamical system, for instance, gives conservation of momentum. Translational invariance of the medium gives conservation of a wave property which may be called *pseudomomentum*. Here I am following established terminology in both plasma and solid-state physics (e.g. Sturrock 1962*b*; Gordon 1973; Peierls 1976, 1979), where the importance of keeping the distinction between momentum and pseudomomentum in mind has long been recognized. In the case of a slowly modulated, small-amplitude wavetrain whose wave-energy density $\hat{E}$ is well defined, in the sense discussed by Bretherton & Garrett (1968), the density of pseudomomentum can be shown to be approximately equal to

$$\hat{E}\mathbf{k}/\hat{\omega}, \tag{5.1}$$

where $\mathbf{k}$ is the wavenumber and $\hat{\omega}$ the intrinsic frequency of the waves, i.e. their frequency relative to the local medium. To the same approximation, the density of pseudoenergy is equal to $\hat{E}\omega/\hat{\omega}$, where ω is the frequency in the observer's frame (thus for a medium at rest the density of pseudoenergy coincides with $\hat{E}$). Exact definitions are given by Andrews & McIntyre (1978*b*). It can again be asked whether momentum and pseudomomentum are related in any way. Once more it turns out that, although there is no general relationship between their densities, there is a fairly close relationship (although again not equality) between their fluxes (Gordon 1973, Andrews & McIntyre 1978*b*, § 5.2). This is the reason why formulae like (2.2) arise.

It is similarly the reason, or part of the reason, why resultant mean forces due to waves, for example the wave-drag force on the towed obstacle in figure 1 (*a*), are often the same *as if*

(*a*) the waves had momentum equal to their pseudomomentum, and
(*b*) the medium were absent.

Let us call this 'the pseudomomentum rule'. The rule turns out to hold in so many cases studied in the laboratory that it has sometimes been mistaken for physical reality (Gordon, *op. cit.*). That is, the important words 'as if' have sometimes been forgotten, together with the distinction between momentum and pseudomomentum.

Whether or not the pseudomomentum rule holds in a given case depends upon the effects of the part of the radiation stress, or wave-induced momentum flux, which is unrelated to the pseudomomentum flux. In the acoustic example of § 2 that part is the isotropic, pressure-like contribution appearing in (2.1), proportional to

$$\partial(\log c)/\partial(\log \rho);$$

and of course the example of § 2 is one in which the pseudomomentum rule does *not* hold, since the isotropic contribution does add to the mean forces on the tube walls. Such contributions arise from nonlinearity in the restoring forces to which the waves owe their existence, and are analogous to the excess mean force arising when a hard spring is alternately stretched and compressed about some mean position. In many cases these contributions to the radiation stress do not cause anything very interesting to happen, since their effects may be nullified, if enough time is available for a balance to be reached, by stresses resulting from small, $O(a^2)$ displacements of the material of the medium. In the case of acoustic waves, for instance, an $O(a^2)$ mean dilatation, such as would occur in the example of § 2 if the walls of the tube were allowed to expand outwards under the influence of the radiation stress, will lead to an adjustment in the mean pressure field which can compensate the isotropic component of the radiation stress. The stresses involved need not be isotropic and pressure-like if the medium *per se* can support anisotropic stresses. An example is the Maxwell stress involved in the restoring forces for Alfvén-acoustic waves (Dewar 1970, eq. 34).

The foregoing remarks suggest, and detailed analysis confirms (Gordon 1973; McIntyre 1981) that in order for the pseudomomentum rule to hold it is generally necessary (albeit not generally sufficient as we shall see)

(i) that changes in the ambient pressure, or in other stresses supportable by the medium *per se* in the absence of waves, should not be able to affect the resultant mean force whose value is of interest, and

(ii) that mean conditions should vary in time sufficiently slowly, if at all, for $O(a^2)$ stresses in the medium to have time to reach equilibrium with the part of the radiation stress which is unrelated to the pseudomomentum flux.

These requirements are fulfilled, for example, by the substantially steady mean forces which generate acoustic streaming of the 'quartz-wind' type (e.g. Eckart 1948; Markham 1952; Westervelt 1953; Nyborg 1953; Lighthill 1978), and by the analogous horizontal mean forces set up by dissipating internal gravity waves, recently verified experimentally by the work of Plumb & McEwan (1978). They are fulfilled also in most experiments on wave-induced mean displacements of interfaces between fluids of differing composition, such as those carried out by Hertz & Mende (1939) for sound waves, and by Ashkin & Dziedzic (1973) for light waves. They are fulfilled in most experiments in which a steady wavetrain is scattered from or absorbed by an immersed obstacle.

Measurements of the mean forces on immersed obstacles have in fact provided the most accurate experimental verifications of the pseudomomentum rule. In the case of sound waves such measurements have been carried out on numerous occasions, since they are used routinely for absolute determinations of the acoustic intensities produced by ultrasonic transducers. Provided that precautions are taken to minimize the effects of acoustic streaming, accuracies of the order of a percent or so are achieved for transducers producing a few milliwatts (e.g. Rooney 1973*a*, *b*). The behaviour of the mean force as a function of frequency, as predicted by the pseudomomentum rule in conjunction with accurate linear calculations of the scattered sound field (see Westervelt 1957), has been confirmed in great detail by Hasegawa & Yosioka (1975) and others. The corresponding experiment for light waves has been done very accurately by Jones & Leslie (1978), improving on earlier measurements by Jones & Richards in 1954. Jones & Leslie verify the pseudomomentum rule 'to a precision of better than 0·1 %'. Longuet-Higgins (1977) has measured the mean forces due to absorption of surface gravity waves by a Cockerell wave raft, and although the experiments were not done at such high precision as the preceding they are good enough to verify the pseudomomentum rule quantitatively. (He also gave an elegant demonstration of a model boat which propelled itself by emitting surface gravity waves to the rear; this is another case in which the rule should hold, but no quantitative experimental check was made.) Other experiments on surface gravity waves either quoted or carried out by Longuet-Higgins (*op. cit.*) are consistent with the rule, provided that due account is taken of all the scattered waves in calculating the net flux, including harmonics generated by nonlinear processes near the obstacle, and provided also that 'resultant mean force' is understood to include forces exerted on the fluid as well as the obstacle, in cases where wave dissipation near the obstacle gives rise to mean streaming.

The correctness of the pseudomomentum rule in so many cases accessible to laboratory experiment appears to have been a major reason for continuation of the disputes over wave 'momentum', in particular the celebrated Abraham–Minkowski controversy over 'the momentum' of light waves in a refractive medium. As has been pointed out by a number of authors, however, notably Penfield & Haus (1966), Gordon (1973) and Peierls (1976), Abraham's momentum is the electromagnetic contribution to the actual momentum, whereas Minkowski's is the pseudomomentum (Gordon *op. cit.*, Peierls 1976, § 6). In the presence of a material medium they are unequal; but there is nothing paradoxical about this once one recognizes that they are distinct physical entities. Similarly, there is nothing paradoxical about the fact that the surface–gravity-wave packet in figure 2 has zero momentum, but pseudomomentum equal to I. This incidentally is another case which conforms to the pseudomomentum rule, not only for generation by towing, but also for reflexion or absorption by an immersed obstacle, as can easily be shown by a slight extension of the analysis of Longuet-Higgins (1977, §2). The change in *momentum* due to the recoil on the obstacle is entirely accounted for by the generation of $O(a^2)$ long-wave transients, as before.

It is perhaps worth mentioning a few more examples where the pseudomomentum rule does not hold. (If the rule represented physical reality, there could be no such examples, of course.) Among those of the obvious 'static' type typified by the problem of § 2 one might mention the thermal expansion of liquids, in which the thermal vibrations themselves can be modelled as sound waves; it was this problem which originally motivated Brillouin's extensive investigation into acoustic radiation stress. A related problem arises in connection with stellar atmospheres permeated by shock waves. The waves must cause departures from hydrostatic balance, a matter of some importance for the interpretation of observed spectral lines. Effects of the general type represented by the $\partial(\log c)/\partial(\log \rho)$ contribution to (2.1) are clearly important in all such problems.

An example of greater fluid-dynamical interest arises in studies of the solar wind. Some of the more successful models take account of the acceleration of the solar wind by Alfvén waves (Hollweg 1978, Jacques 1978, and references). The mean flow is so fast that the mean pressure and Maxwell stress have no time to equilibrate. Inevitably all contributions to the radiation stress are significant, not just the contribution which equals the pseudomomentum flux. The same is true of sound waves on a trans-sonic mean flow, such as that in a rocket nozzle (Jacques 1977), or of surface gravity waves on a weir.

The 'parametric acoustic array' (Westervelt 1963), in which the radiation stress due to a beam of high-frequency sound waves of fluctuating amplitude acts directly on the medium to generate a narrow beam of low-frequency sound waves, provides another example in which the mean dynamics (that of the low frequency sound) is too fast for equilibration. The isotropic, $\partial(\log c)/\partial(\log \rho)$ contribution to the acoustic radiation stress (2.1) plays an important role, indeed in many cases a dominant one since for most liquids $\partial(\log c)/\partial(\log \rho)$ is considerably greater than unity. There would be gross disagreement with experiment if that contribution were omitted (e.g. Moffett *et al.* 1971). This is interesting in itself since at present, as far as I know, experiments on parametric arrays provide the only direct experimental check on the isotropic contribution to (2.1), a fact which does not seem to have been pointed out before.†

† *Note added in proof*: Westervelt (1977) mentions the connexion.

A completely different type of exception to the pseudomomentum rule arises for waves in a stratified fluid. If an isolated packet of internal gravity waves such as those shown in figure 1 (*a*) is scattered by a two-dimensional barrier immersed in the fluid, then the mean horizontal force on the barrier is quite different (in sign as well as magnitude, in some cases) from that given by the pseudomomentum rule. I pointed this out in my (1973) paper. In that case there is plenty of time for equilibration, condition (ii) above, but it turns out that in order for the rule to hold for an obstacle immersed in a stratified fluid under gravity the obstacle must be three-dimensional, so that a closed curve can be drawn around the obstacle on each isentropic surface. Even then the rule holds only for the horizontal components of the mean force. The full details, to be given in my book (1981), would take us too far astray here.

6. Concluding remarks

The reader will have noticed, or guessed from (5.1), that pseudomomentum and its temporal analogue, pseudoenergy, are closely related to the notion of wave-action. Essentially the same formalism underlies the conservation relations for all three quantities. As Andrews and I pointed out in our (1978*b*) paper, that formalism is mathematically (not physically!) identical to the formalism associated with the energy-momentum tensor in classical field theory (Landau & Lifshitz 1975), as was shown in effect by the work of Sturrock (1962*a*, *b*) and Hayes (1970). The symmetry condition required for wave-action conservation is invariance of the medium to a phase shift in the wave field, a condition which holds tautologically if mean quantities are defined by averaging over phase. This type of averaging occurs naturally whenever the method of multiple scales is used as the basis for distinguishing waves from mean state (Whitham 1970, 1974). Phase invariance thus leads to a conservation relation (e.g. Sturrock 1962*a*; Whitham 1965; Bretherton & Garrett 1968; Hayes 1970) even when the medium is time-dependent and inhomogeneous in all spatial directions. The phase is of course a physically real, observable phase, quite unlike the unobservable phases involved in the mathematically analogous relations in quantum mechanics, which are the conservation relations expressing, where appropriate, conservation of the number of quanta.

It was suggested earlier that the issues involved in the wave momentum myth are essentially ones of classical physics, a fact which should be plain enough from the discussion. However, a final remark about quantum mechanics may be in order. It is noticeable that the myth has a tendency to reappear in the fluid-dynamical literature whenever ideas or computational techniques are borrowed from quantum mechanics. Quantization carries suggestions of 'particle-like' behaviour, leading perhaps to a tacit presumption that the quanta behave in every way like isolated particles in a vacuum. That this presumption cannot be correct when a material medium is present is obvious from what has already been said. It remains just as true as before that, when a material medium is present, there are two distinct kinds of translational symmetry. Symmetries correspond to conservable quantities in quantum mechanics just as much as in classical mechanics, and so the two symmetries in question still give rise to two physically distinct conservable quantities, momentum and pseudomomentum.

To prevent the same old disputes dragging on for another half-century, only one thing is needed: the widespread acceptance of a terminology which explicitly recognizes the fact that pseudomomentum and momentum are not the same thing. The term pseudomomentum is now well established in other branches of physics, and it seems to me that the only sensible course is to adopt it in our own.

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REFERENCES

ANDREWS, D. G. 1974 Mean flows forced by waves in stratified and rotating fluids. Ph.D. thesis, University of Cambridge.

ANDREWS, D. G. & McINTYRE, M. E. 1978*a* An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.* **89**, 609–646.

ANDREWS, D. G. & McINTYRE, M. E. 1978*b* On wave-action and its relatives. *J. Fluid Mech.* **89**, 647–664.

ASHKIN, A. & DZIEDZIC, J. M. 1973 Radiation pressure on a free liquid surface. *Phys. Rev. Lett.* **30**, 139–142.

BAINES, P. G. 1979 Observations of stratified flow over two-dimensional obstacles in fluid of finite depth. *Tellus* **31**, 351–371.

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press, 615 pp.

BENJAMIN, T. B. 1970 Upstream influence. *J. Fluid Mech.* **40**, 49–79.

BONDI, H. 1967 *Assumption and Myth in Physical Theory*. Cambridge University Press, 88 pp.

BRETHERTON, F. P. 1969 On the mean motion induced by internal gravity waves. *J. Fluid Mech.* **36**, 785–803.

BRETHERTON, F. P. 1971 The general linearised theory of wave propagation, §6. *Lectures in Applied Mathematics* (American Math. Soc.), **13**, 61–102.

BRETHERTON, F. P. & GARRETT, C. J. R. 1968 Wavetrains in inhomogeneous moving media. *Proc. Roy. Soc.* A **302**, 529–554.

BRILLOUIN, L. 1925 On radiation stresses (in French). *Annales de Physique* **4**, 528–586.

CARROLL, L. 1871 *Through the Looking-Glass*. London; Macmillan.

DEWAR, R. L. 1970 Interaction between hydromagnetic waves and a time-dependent inhomogeneous medium. *Phys. Fluids* **13**, 2710–2720.

DEWAR, R. L. 1977 Energy-momentum tensors for dispersive electromagnetic waves. *Aust. J. Phys.* **30**, 533–575.

DYSTHE, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. Roy. Soc. London* A **369**, 105–114.

ECKART, C. 1948 Vortices and streams caused by sound waves. *Phys. Rev.* **73**, 68–76.

FUBINI-GHIRON, E. 1937 La tension de radiation acoustique et les ondes de grande amplitude. *Rev. d'Acoustique* **6**, 69.

GORDON, J. P. 1973 Radiation forces and momenta in dielectric media. *Phys. Rev.* A **8**, 14–21.

HASEGAWA, T. & YOSIOKA, J. 1975 Acoustic radiation force on fused silica spheres and intensity determination. *J. Acoust. Soc. Amer.* **58**, 581–585.

HAYES, W. D. 1970 Conservation of action and modal wave action. *Proc. Roy. Soc.* A **320**, 187–208.

HERTZ, G. & MENDE, H. 1939 Der Schallstrahlungsdruck in Flüssigkeiten. *Z. Phys.* **114**, 354–367.

Hollweg, J. V. 1978 Some physical processes in the solar wind. *Revs Geophys. & Space Phys.* **16**, 689–720.

Israel, W. 1977 Relativistic effects in dielectrics: an experimental decision between Abraham and Minkowski? *Phys. Lett.* **67** B, 125–128.

Jacques, S. A. 1977 Momentum and energy transport by waves in the solar atmosphere and solar wind. *Astrophys. J.* **215**, 942–951.

Jacques, S. A. 1978 Solar wind models with Alfvén waves. *Astrophys. J.* **226**, 632–649.

Jones, R. V. & Leslie, B. 1978 The measurement of optical radiation pressure in dispersive media. *Proc. Roy. Soc.* A **360**, 347–363.

Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press, 738 pp.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics*, 1st Engl. ed. Pergamon, 536 pp.

Landau, L. D. & Lifshitz, E. M. 1975 *The Classical Theory of Fields*, 4th revised Engl. ed. Pergamon, 402 pp.

Lighthill, M. J. 1978 Acoustic streaming. *J. Sound Vib.* **61**, 391–418.

Longuet-Higgins, M. S. 1977 The mean forces exerted by waves on floating or submerged bodies with applications to sand bars and wave power machines. *Proc. Roy. Soc. Lond.* A **352**, 463–480.

Longuet-Higgins, M. S. & Stewart, R. W. 1962 Radiation stress nd mass transport in gravity waves, with application to 'surf beats'. *J. Fluid Mech.* **13**, 481–504.

Markham, J. J. 1952 Second-order acoustic fields: streaming with viscosity and relaxation. *Phys. Rev.* **86**, 497–502.

McIntyre, M. E. 1972 On Long's hypothesis of no upstream influence in uniformly stratified or rotating flow. *J. Fluid Mech.* **52**, 209–243.

McIntyre, M. E. 1973 Mean motions and impulse of a guided internal gravity wave packet. *J. Fluid Mech.* **60**, 801–811.

McIntyre, M. E. 1981 *Waves and Mean Flows.* Monograph in preparation.

Moffett, M. B., Westervelt, P. J. & Beyer, R. T. 1971 Large-amplitude pulse propagation – a transient effect. II. *J. Acoust. Soc. Amer.* **49**, 339–343.

Nyborg, W. L. 1953 Acoustic streaming due to attenuated plane waves. *J. Acoust. Soc. Amer.* **25**, 68–75.

Peierls, R. E. 1976 The momentum of light in a refracting medium. *Proc. Roy. Soc.* A **347**, 475–491.

Peierls, R. E. 1979 *Surprises in Theoretical Physics.* Princeton University Press, 166 pp.

Penfield, P. & Haus, H. A. 1966 Hamilton's principle for electromagnetic fields. *Phys. Fluids* **9**, 1195–1204.

Post, E. J. 1953 Radiation pressure and dispersion. *J. Acoust. Soc. Amer.* **25**, 55–60.

Plumb, R. A. & McEwan, A. D. 1978 The instability of a forced standing wave in a viscous, stratified fluid: a laboratory analogue of the quasi-biennial oscillation. *J. Atmos. Sci.* **35**, 1827–1839.

Rayleigh, Lord 1902 On the pressure of vibrations. *Phil. Mag.* **3**, 338–346 [also *Sci. Papers* **5**, 41–48].

Rayleigh, Lord 1905 On the momentum and pressure of gaseous vibrations, and on the connexion with the virial theorem. *Phil. Mag.* **10**, 364–374. [*Scientific Papers* **5**, 262–271.]

Robinson, F. N. H. 1975 Electromagnetic stress and momentum in matter. *Physics Reports* (Sec. C of *Physics Letters*) **16**, 314–354.

Rooney, J. A. 1973*a* Does radiation pressure depend on B/A? *J. Acoust. Soc. Amer.* **54**, 429–430. [Note that B/A means $2\partial(\log c)/\partial(\log \rho)$.]

Rooney, J. A. 1973*b* Determination of acoustic power outputs in the microwatt–milliwatt range. *Ultrasound in Med. & Biol.* **1**, 13–16.

Sturrock, P. A. 1962*a* Field-theory analogs of the Lagrange and Poincaré invariants. *J. Mathematical Physics* **3**, 43–50. (§§VI, VII.)

Sturrock, P. A. 1962*b* Energy and momentum in the theory of waves in plasmas. *Plasma Hydromagnetics* (Lockheed Symposium, ed. D. Bershader), pp. 47–57. Stanford U.P. (equation (4.7)).

WESTERVELT, P. J. 1953 The theory of steady rotational flow generated by a sound field. *J. Acoust. Soc. Amer.* **25**, 60–67.

WESTERVELT, P. J. 1957 Acoustic radiation pressure. *J. Acoust. Soc. Amer.* **29**, 26–29.

WESTERVELT, P. J. 1963 Parametric acoustic array. *J. Acoust. Soc. Amer.* **35**, 535–537.

WESTERVELT, P. J. 1977 Sound. *McGraw Hill Yearbook Science & Technology* (ed. D. N. Lapedes) **16**, 389–390.

WHITHAM, G. B. 1965 A general approach to linear and nonlinear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273–283.

WHITHAM, G. B. 1970 Two-timing, variational principles and waves. *J. Fluid Mech.* **44**, 373–395.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley. Interscience, 636 pp.